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Analytical Investigation of Generalized Gauss Hypergeometric Functions Using Finite Integral Formulae

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ABSTRACT

The generalized Gauss hypergeometric function constitutes one of the most fundamental classes of special functions due to its rich analytical structure and its extensive applications in mathematical analysis, differential equations, mathematical physics, and applied sciences. The function serves as a unifying framework for numerous elementary and higher transcendental functions while providing effective analytical tools for investigating integral representations, transformation identities, convergence properties, and parameter-dependent behaviour. Consequently, the analytical investigation of generalized Gauss hypergeometric functions continues to attract considerable attention in modern special function theory.

The present paper investigates the generalized Gauss hypergeometric function by employing finite integral formulae together with classical identities involving the Beta and Gamma functions. A systematic analytical framework is developed beginning with the mathematical preliminaries of the generalized Gauss hypergeometric function, followed by the derivation of finite integral representations under suitable parameter restrictions. The analytical properties of these representations are subsequently examined with particular emphasis on parameter behaviour, convergence characteristics, transformation formulae, and analytical applications. Furthermore, the close relationship between finite integral representations and generalized Gauss hypergeometric functions is established through the Euler-type integral approach.

The results demonstrate that finite integral representations provide an efficient analytical technique for investigating generalized Gauss hypergeometric functions and their associated transformation identities. The mathematical developments presented in this paper contribute to the analytical theory of generalized hypergeometric functions and provide a useful foundation for further investigations involving special functions, integral transforms, and mathematical physics.

[Keywords: Generalized Gauss Hypergeometric Function; Finite Integral Formulae; Euler Integral; Beta Function; Gamma Function; Transformation Formulae; Mathematical Analysis; Special Functions.]

1. Introduction

The Gauss hypergeometric function occupies a distinguished position in the theory of special functions because of its remarkable analytical structure and its ability to represent a wide variety of elementary and higher transcendental functions within a unified mathematical framework. As a particular case of the generalized hypergeometric function, the Gauss hypergeometric function has been extensively investigated owing to its close relationship with differential equations, integral representations, orthogonal polynomials, conformal mappings and numerous problems arising in mathematical physics.[1,2]

The analytical theory of the Gauss hypergeometric function originated from the pioneering investigations of Euler and Gauss, whose contributions established the fundamental relationship between infinite series, differential equations and definite integrals. These classical developments were subsequently extended by Bailey, Slater, Erdélyi and several other researchers, leading to a comprehensive theory of generalized hypergeometric functions and their applications in pure and applied mathematics.[1–4] The resulting analytical framework continues to play a significant role in modern investigations involving integral transforms, asymptotic analysis, fractional calculus and special function theory.[5,6]

Among the various analytical techniques available for the study of hypergeometric functions, finite integral representations constitute one of the most effective approaches. Such representations establish direct connections between generalized hypergeometric series and the classical Beta and Gamma functions, thereby providing elegant analytical methods for deriving transformation identities, reduction formulae and convergence properties. Compared with the defining power-series representation, finite integral formulae frequently provide a more convenient analytical framework for investigating parameter-dependent behaviour and functional transformations.[3,7]

The generalized Gauss hypergeometric function possesses several important analytical properties that distinguish it from the broader class of generalized hypergeometric functions. Its finite integral representation serves not only as an alternative functional expression but also as a powerful analytical tool for investigating convergence, parameter restrictions and transformation properties. Furthermore, the intimate relationship between finite integral representations and Euler-type integrals provides a systematic procedure for analysing generalized Gauss hypergeometric functions under suitable mathematical conditions.[2,8]

In recent years, generalized hypergeometric functions have found extensive applications in approximation theory, mathematical physics, quantum mechanics, fractional differential equations, integral transforms and computational mathematics. Their occurrence in diverse branches of mathematics has stimulated renewed interest in developing analytical techniques capable of simplifying their investigation and extending their applicability. Finite integral methods have proved particularly useful in this respect because they transform infinite series into analytically tractable integral forms while preserving the essential mathematical characteristics of the original functions.[6,9]

The principal objective of the present investigation is to examine the generalized Gauss hypergeometric function through finite integral formulae and their associated analytical properties. Particular emphasis is placed on the derivation of finite integral representations, the study of convergence behaviour, parameter transformations and analytical applications. The mathematical developments presented in this paper provide a unified analytical framework for investigating generalized Gauss hypergeometric functions and establish a foundation for further studies involving transformation identities, special functions and related areas of mathematical analysis.[1,2,6]

The remainder of this paper is organized as follows. Section 2 presents the mathematical preliminaries required for the subsequent investigation. Section 3 develops the finite integral representation of the generalized Gauss hypergeometric function. Section 4

examines the principal analytical properties obtained from the finite integral formulation. Section 5 discusses selected mathematical applications, while the final section summarizes the principal findings and indicates possible directions for future research.

2. Mathematical Preliminaries

This section presents the fundamental definitions, notation and analytical identities required for the development of the finite integral representations discussed in the subsequent sections. Only those mathematical concepts that are directly employed in the analytical investigation of the generalized Gauss hypergeometric function are included. The definitions and identities presented herein constitute the mathematical foundation for deriving finite integral formulae and analysing their associated transformation properties.[1,2]

2.1 Generalized Gauss Hypergeometric Function

The generalized Gauss hypergeometric function is defined by the infinite series

$$ {}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}, $$

where

$$ c \neq 0, -1, -2, \ldots $$

and the denominator parameter is restricted to avoid singularities. The above series converges absolutely for

$$ |z| < 1 $$

while its behaviour on the boundary depends upon the admissible values of the parameters.[1,2]

2.2 Pochhammer Symbol

The coefficients of the generalized Gauss hypergeometric series are expressed in terms of the Pochhammer symbol (or shifted factorial), defined by

$$ (a)_0 = 1 $$

and

$$ (a)_n = a(a+1)(a+2)\cdots(a+n-1), \qquad n \geq 1 $$

An equivalent representation in terms of the Gamma function is

$$ (a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} $$

The Pochhammer symbol plays a central role in the analytical theory of generalized hypergeometric functions because it determines the coefficients of the defining series and facilitates the derivation of transformation identities and finite integral representations.[2,4]

2.3 Euler Beta Function

The Euler Beta function is defined by

$$ B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\, dt, \qquad \Re(x) > 0,\ \Re(y) > 0 $$

The Beta function is related to the Gamma function through the classical identity

$$B(x,y)=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

This relationship is fundamental in deriving finite integral representations of generalized Gauss hypergeometric functions and will be employed repeatedly in the subsequent analytical developments.[3,10]

2.4 Generalized Binomial Expansion

For

$$|x|<1$$

the generalized binomial theorem is given by

$$(1-x)^{-a}=\sum_{n=0}^{\infty}\frac{(a)_n}{n!}x^n$$

This expansion establishes the analytical connection between Euler-type finite integrals and generalized Gauss hypergeometric series. It therefore constitutes one of the principal mathematical tools employed in the derivation of the finite integral formulae presented in the next section.[4,5]

3. Finite Integral Representation of the Generalized Gauss Hypergeometric Function

Finite integral representations constitute one of the most effective analytical techniques for investigating the Gauss hypergeometric function. In contrast to the defining power-series representation, finite integral formulae establish direct relationships between generalized Gauss hypergeometric functions and the classical Beta and Gamma functions. Consequently, they provide elegant analytical methods for studying convergence, parameter transformations and several functional identities. Moreover, finite integral representations simplify many analytical computations and form the basis for numerous applications in special function theory and mathematical physics.[1,2]

Consider the Euler-type integral

Equation (3.1)

$$I(z)=\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}\,dt,$$

subject to the conditions

$$\Re(c)>\Re(b)>0,\qquad |z|<1$$

Using the generalized binomial theorem,

Equation (3.2)

$$(1-zt)^{-a}=\sum_{n=0}^{\infty}\frac{(a)_n}{n!}(zt)^n,\qquad |zt|<1.$$

which is valid under the prescribed convergence condition

Substituting Equation (3.2) into Equation (3.1), we obtain

Equation (3.3)

$$I(z) = \int_0^1 t^{b-1}(1-t)^{c-b-1} \sum_{n=0}^{\infty} \frac{(a)_n}{n!}(zt)^n \, dt.$$

Since the series converges uniformly for |z|<1|z|<1|z|<1, the order of summation and integration may be interchanged. Hence,

Equation (3.4)

$$I(z) = \sum_{n=0}^{\infty} \frac{(a)_n z^n}{n!} \int_0^1 t^{b+n-1}(1-t)^{c-b-1} \, dt.$$

The integral appearing in Equation (3.4) is recognized as Euler's Beta integral. Therefore,

Equation (3.5)

$$\int_0^1 t^{b+n-1}(1-t)^{c-b-1} \, dt = B(b+n, c-b).$$

Applying the classical Beta–Gamma relation to Equation (3.5), we obtain

Equation (3.6)

$$B(b+n, c-b) = \frac{\Gamma(b+n)\Gamma(c-b)}{\Gamma(c+n)}.$$

Substituting Equation (3.6) into Equation (3.4) gives

Equation (3.7)

$$I(z) = \Gamma(c-b) \sum_{n=0}^{\infty} \frac{(a)_n \Gamma(b+n)}{\Gamma(c+n)} \frac{z^n}{n!}.$$

The Gamma function satisfies the recurrence identity

Equation (3.8)

$$\Gamma(b+n) = \Gamma(b)(b)_n.$$

Similarly,

Equation (3.9)

$$\Gamma(c+n) = \Gamma(c)(c)_n.$$

Substituting Equations (3.8) and (3.9) into Equation (3.7), the integral assumes the form

Equation (3.10)

$$I(z) = \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)} \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}.$$

Equation (3.10) establishes the direct analytical connection between the Euler-type finite integral and the defining series of the Gauss hypergeometric function. The result demonstrates that the finite integral representation is completely determined by the Beta and Gamma function identities together with the Pochhammer symbols. This relationship provides the mathematical foundation for the analytical investigations presented in the subsequent sections.[2,3,6]

Recognizing the infinite series appearing in Equation (3.10) as the defining series of the Gauss hypergeometric function, the finite integral immediately assumes the following analytical representation.

Equation (3.11)

$$I(z) = \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)}\, {}_2F_1(a,b;c;z).$$

Solving Equation (3.11) for the generalized Gauss hypergeometric function yields the classical Euler-type finite integral representation.

Equation (3.12)

$${}_2F_1(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}\, dt,$$

subject to

$$\Re(c) > \Re(b) > 0, \qquad |z| < 1.$$

Equation (3.12) establishes an explicit analytical relationship between the defining hypergeometric series and its corresponding Euler-type finite integral. Unlike the power-series representation, the finite integral formulation expresses the generalized Gauss hypergeometric function in terms of classical special functions and provides an alternative analytical framework that is particularly useful for investigating parameter transformations, convergence properties and integral identities.[1,2,3]

Furthermore, the integral representation demonstrates that the analytical behaviour of the generalized Gauss hypergeometric function is governed jointly by the Beta function, the Gamma function and the Pochhammer coefficients. This unified formulation considerably simplifies several analytical investigations and forms the basis for many transformation formulae and reduction identities encountered in the theory of special functions.[4,6]

The finite integral representation also provides an effective analytical technique for studying parameter restrictions and functional behaviour near the boundary of the convergence region. Consequently, it serves as one of the principal mathematical tools employed in modern investigations involving generalized hypergeometric functions and their applications in mathematical analysis and mathematical physics.[2,9,10]

Remark 3.1

The finite integral representation obtained in Equation (3.12) constitutes one of the fundamental analytical formulations of the generalized Gauss hypergeometric function. Throughout the remainder of this paper, this representation will serve as the principal analytical tool for investigating the behaviour of the function under various parameter conditions and for discussing its applications in special function theory.

4. Analytical Investigation

The finite integral representation derived in the preceding section provides an effective analytical framework for investigating the mathematical behaviour of the generalized Gauss hypergeometric function. Unlike the defining power-series representation, the integral formulation establishes a direct connection between the Gauss hypergeometric function and the classical Beta and Gamma functions. Consequently, several important analytical properties concerning parameter restrictions, convergence behaviour and functional transformations can be investigated in a systematic manner.[1,2]

4.1 Behaviour of the Parameters

The analytical behaviour of the generalized Gauss hypergeometric function depends significantly upon the parameters aaa, bbb and ccc. The numerator parameters aaa and bbb determine the coefficients of the hypergeometric series through the Pochhammer symbols, whereas the denominator parameter ccc governs the normalization of the coefficients and influences the convergence characteristics of the series. Consequently, appropriate restrictions on these parameters are essential for ensuring the existence of both the series representation and the corresponding finite integral representation.[2,4]

From Equation (3.12), it is evident that the Beta integral exists only when

Equation (4.1)

$$\Re(c) > \Re(b) > 0.$$

The above condition guarantees the convergence of the Euler-type finite integral and simultaneously ensures that the Gamma functions occurring in the normalization constant remain finite.[3,10]

4.2 Convergence Behaviour

The generalized Gauss hypergeometric series converges absolutely within the unit disc of the complex plane. Therefore,

Equation (4.2)

$$|z| < 1.$$

Within this region, both the defining series and the finite integral representation describe the same analytic function. The finite integral representation, however, frequently provides a more convenient analytical expression for investigating limiting behaviour and parameter transformations than the corresponding infinite series.[1,2,5]

4.3 Analytical Significance of the Finite Integral Representation

The finite integral representation developed in Section 3 establishes a direct analytical relationship between generalized Gauss hypergeometric functions and the classical Beta and Gamma functions. This relationship considerably simplifies the derivation of analytical identities because the integral kernel possesses well-established convergence and transformation properties. Consequently, finite integral techniques have become one of the principal analytical methods employed in modern investigations involving generalized hypergeometric functions.[4,6]

Furthermore, the integral representation provides a unified mathematical framework in which the defining hypergeometric series, Euler's Beta integral and the Gamma function are connected through a single analytical expression. Such a formulation facilitates theoretical investigations and provides greater flexibility in studying functional behaviour under different parameter conditions.[5,7]

4.4 Analytical Observations

The analytical investigation presented above leads to the following observations:

1. The generalized Gauss hypergeometric function admits equivalent power-series and finite integral representations.

2. The Euler-type finite integral representation is valid under the parameter restriction $\Re(c) > \Re(b) > 0$.

3. The Beta and Gamma functions constitute the principal analytical tools for establishing the finite integral representation.

4. The finite integral formulation provides an effective alternative to the defining series for investigating convergence and parameter-dependent behaviour.

5. The analytical framework developed in the present investigation forms a suitable basis for studying transformation identities and applications involving generalized Gauss hypergeometric functions.[1,2,6]

Remark 4.1

The analytical investigation confirms that finite integral methods provide a mathematically consistent and computationally convenient framework for studying generalized Gauss hypergeometric functions. The close relationship between Euler-type integrals, Beta functions and hypergeometric series illustrates the unified analytical structure underlying the theory of special functions and forms the basis for the applications discussed in the following section.

5. Applications of the Generalized Gauss Hypergeometric Function

The generalized Gauss hypergeometric function possesses a wide range of applications in both pure and applied mathematics owing to its rich analytical structure and its close relationship with numerous classes of special functions. The finite integral representation established in Section 3 provides an efficient analytical framework for investigating mathematical models involving differential equations, integral transforms and higher transcendental functions. Consequently, generalized Gauss hypergeometric functions continue to play an important role in contemporary mathematical analysis.[1,2]

5.1 Applications in Differential Equations

One of the most significant applications of the generalized Gauss hypergeometric function arises in the analytical solution of second-order linear differential equations having regular singular points. Numerous equations encountered in mathematical physics, applied mathematics and engineering may be transformed into the Gauss hypergeometric differential equation through appropriate substitutions. The finite integral representation developed in the present investigation provides an alternative analytical formulation that complements the classical power-series solution and facilitates the study of parameter-dependent solutions.[1,2,4]

5.2 Applications in Integral Transforms

Finite integral representations establish a direct relationship between generalized Gauss hypergeometric functions and classical integral transforms such as the Mellin and Laplace transforms. Such relationships simplify the evaluation of definite integrals and provide alternative analytical expressions for functions represented by infinite series. Furthermore, the Beta–Gamma identities incorporated in the finite integral formulation enable several transformation techniques that are widely employed in asymptotic analysis and applied mathematical modelling.[6,8]

5.3 Applications in Special Function Theory

Many classical special functions, including Legendre functions, Chebyshev polynomials, Jacobi polynomials and several orthogonal polynomial systems, may be expressed in terms of the Gauss hypergeometric function. Consequently, analytical properties established for the generalized Gauss hypergeometric function can frequently be extended to these related classes of special functions. The finite integral representation therefore provides a unified analytical framework for investigating numerous mathematical identities and functional relationships.[3,5]

5.4 Applications in Mathematical Physics

Generalized Gauss hypergeometric functions occur naturally in several branches of mathematical physics, including quantum mechanics, wave propagation, potential theory and electromagnetic field analysis. In these areas, analytical solutions of governing differential equations are frequently represented by hypergeometric functions. The finite integral representation derived in this investigation provides an alternative analytical approach for studying such solutions and contributes to a deeper understanding of their mathematical structure.[2,6,9]

5.5 Concluding Remarks

The applications discussed above demonstrate the analytical versatility of the generalized Gauss hypergeometric function and highlight the importance of finite integral representations in contemporary mathematical analysis. By establishing explicit relationships between Euler-type integrals, Beta and Gamma functions and hypergeometric series, the finite integral formulation provides a unified mathematical framework that supports theoretical investigations as well as practical applications in several branches of mathematics and mathematical physics.

6. Conclusion

The present study investigated the generalized Gauss hypergeometric function through finite integral formulae and their associated analytical properties. Beginning with the fundamental mathematical preliminaries, a finite integral representation was derived by employing Euler-type integrals together with the classical Beta and Gamma function identities. The resulting formulation established a direct analytical relationship between the power-series definition and the corresponding finite integral representation, providing an alternative framework for studying the generalized Gauss hypergeometric function.

The analytical investigation demonstrated that finite integral representations offer an effective approach for examining parameter behaviour, convergence characteristics and functional relationships. Furthermore, the applications discussed in this study illustrate the usefulness of the generalized Gauss hypergeometric function in differential equations, integral transforms, special function theory and mathematical physics. The finite integral approach simplifies several analytical procedures while preserving the fundamental mathematical properties of the function.

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